Construction of spinors in various dimensions

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These notes grew out of a desire to have a nice Majorana representation of the gamma matrices in eight Euclidean dimensions. I failed to obtain this by guesswork, so had to approach it systematically, by induction from two dimensions with a few tricks along the way. As such, I ended up with a nice tour through Euclidean spinors in dimension up to eight. There are also some brief comments about what changes in Lorentzian signature.

1 Dirac, Weyl, Majorana

One thing which confused me for quite some time as a beginning graduate student was the distinction between 'Dirac', 'Weyl' and 'Majorana' spinors. I think this was because of the slightly unusual usage of these terms in physics parlance; mathematically, it is reasonably straightforward.

I will mostly discuss spinors in Euclidean signature i.e. spinor representations of $SO(D)$ rather than $SO(D-1,1)$, so that our gamma matrices will simply be Hermitian, and satisfy

$$\{\gamma^m, \gamma^n\} = 2\delta^{mn} \ . \tag{1}$$

In any dimension, we can define a $2^{\lfloor \frac{D}{2} \rfloor}$-dimensional unitary representation of $\mathrm{Spin}(D)$ by taking generators

$$\sigma^{mn} = \frac{1}{4}[\gamma^m, \gamma^n] \ . \tag{2}$$

These are automatically anti-Hermitian since the gamma matrices are Hermitian, so indeed we have that $e^{t_{mn}\sigma^{mn}} \in SU(2^{\lfloor \frac{D}{2} \rfloor})$.

If the gamma matrices can be chosen to be either real or purely imaginary, then the generators Equation (2) will be real, and the image of $\mathrm{Spin}(D)$ will actually lie in $SO(2^{\lfloor \frac{D}{2} \rfloor}) \subset SU(2^{\lfloor \frac{D}{2} \rfloor})$. If this is the case, we can impose that our spinors are real-valued; they are then called Majorana spinors. Any more general choice of gamma matrices is related to the Majorana representation by a similarity transformation, and the corresponding change of basis will typically not preserve the reality of the spinors. However, we can ask what the image of the real spinors is under such a change of basis, and this real subspace will still be preserved by $\mathrm{Spin}(D)$ transformations. This boils down to the following: for a general representation of the gamma matrices, the Majorana spinors are those which satisfy $\psi = C\psi^*$, where C is constructed from the gamma matrices. We will see an explicit example in Section 2.4.

If D is even, we can define the *chirality operator*

$$\hat{\gamma} = \mathrm{i}^{\pm\frac{D}{2}} \prod_{m=1}^{D} \gamma^m \ .$$

where the sign of the exponent can be chosen for convenience. This is Hermitian, and satisfies $\hat{\gamma}^2 = \mathbf{1}$, so has eigenvalues ± 1. It also anti-commutes with all the gamma matrices, $\{\hat{\gamma}, \gamma^m\} = 0$, and so commutes with the spin generators σ^{mn}. The action of $\mathrm{Spin}(D)$ therefore preserves the eigenspaces of $\hat{\gamma}$, indicating that the Dirac representation splits into two irreducible representations, the left- and right-handed *Weyl spinor* representations.

Weyl spinors do not exist in odd dimensions, because if we make a similar definition of $\hat{\gamma}$, we find that it commutes with all the gamma matrices. Since they generate the relevant matrix algebra, this implies that $\hat{\gamma}$ is just a multiple of the identity. So in odd dimensions, the Dirac representation is irreducible.

2 Spinors in various dimensions

One straightforward, and instructive, way to build spinor representations in various dimensions is to start with $D = 2$ and work inductively.

2.1 $D = 2$

Let us start in two dimensions, where we can 'solve' (1) by relatively straightforward guesswork:

$$\gamma^1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \ , \ \gamma^2 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \ .$$

Then the single generator of $\mathrm{Spin}(2)$ is given by

$$\sigma^{12} = \frac{1}{2} \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \ ,$$

which is the standard generator of $SO(2)$ multiplied by a factor of $\frac{1}{2}$, corresponding to the characteristic spinor behaviour of picking up a factor of -1 under rotation by 2π. Note that I deliberately chose real gamma matrices, showing that we can take our spinors to be real i.e. Majorana.

Since we are in an even number of dimensions, we can define the chirality operator

$$\hat{\gamma} = \mathrm{i}\gamma^1\gamma^2 = \begin{pmatrix} 0 & -\mathrm{i} \\ \mathrm{i} & 0 \end{pmatrix} \ .$$

It is trivial to see that the eigenvectors of this operator, which are the Weyl spinors, must be complex-valued, so Majorana-Weyl spinors do not exist for $SO(2)$.

Note that we could alternatively have picked the gamma matrices to be

$$\gamma^1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad , \; \gamma^2 = \begin{pmatrix} 0 & \mathrm{i} \\ -\mathrm{i} & 0 \end{pmatrix} \; ,$$

in which case we find

$$\sigma^{12} = \frac{1}{2}\begin{pmatrix} -\mathrm{i} & 0 \\ 0 & \mathrm{i} \end{pmatrix} \; , \; \hat{\gamma} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \; .$$

So in this case, the two components of our spinor simply transform under $Spin(2)$ with equal and opposite phases, and correspond to the two one-dimensional Weyl representations.

Geometrically, the case $D = 2$ is special, since $SO(2) \cong U(1)$ has an infinite fundamental group, $\pi_1\big(SO(2)\big) \cong \mathbb{Z}$. In all higher dimensions, $\pi_1\big(SO(D)\big) \cong \mathbb{Z}_2$, and $\mathrm{Spin}(D)$ is the unique covering group. When $D = 2$, the group $\mathrm{Spin}(2)$ is just isomorphic to $SO(2)$ itself, but geometrically corresponds to 'unwrapping' it once i.e. it is the double cover associated with the subgroup $2\mathbb{Z} \subset \mathbb{Z}$.

2.2 $D = 3$

Going from $D = 2N$ to $D = 2N+1$ is always trivial; we get the extra gamma matrix for free in the form of the chirality operator $\hat{\gamma}$ from the dimension below. Since the Pauli matrices generate the Clifford algebra for $SO(3)$, we will use slightly different notation in this case, and take our 'gamma' matrices to be:

$$\sigma^1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \; , \; \sigma^2 = \begin{pmatrix} 0 & -\mathrm{i} \\ \mathrm{i} & 0 \end{pmatrix} \; , \; \sigma^3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \; .$$

As already discussed, there are no Weyl representations in odd dimensions. This representation of the gamma matrices does not manifestly allow Majorana spinors, but could we change basis to make them all real or purely imaginary? The answer is obviously no, because the above matrices (and hence their commutators) generate $\mathfrak{su}(2)$. So $\mathrm{Spin}(3) \cong SU(2)$, which has no real representations.

To see why there are no Majorana spinors for $\mathrm{Spin}(3)$ without relying on prior knowledge, one can construct the charge conjugation operator and see that it squares to minus the identity.

We can skip the other odd dimensions, since in each case the gamma matrices are just those of one dimension lower, along with the chirality operator from that dimension.

2.3 $D = 4$

Passing from $D = 2N - 1$ to $D = 2N$ is only marginally more complicated than the even-to-odd case. For $D = 4$, we can build our gamma matrices out blocks of the $D = 3$ gamma matrices given above, as follows:

$$\gamma^1 = \begin{pmatrix} 0 & \mathrm{i}\sigma^1 \\ -\mathrm{i}\sigma^1 & 0 \end{pmatrix}, \quad \gamma^2 = \begin{pmatrix} 0 & \mathrm{i}\sigma^2 \\ -\mathrm{i}\sigma^2 & 0 \end{pmatrix},$$

$$\gamma^3 = \begin{pmatrix} 0 & \mathrm{i}\sigma^3 \\ -\mathrm{i}\sigma^3 & 0 \end{pmatrix}, \quad \gamma^4 = \begin{pmatrix} 0 & \mathbf{1} \\ \mathbf{1} & 0 \end{pmatrix},$$

where $\mathbf{1}$ is the 2×2 identity matrix. It is easy to see how to generalise this to $D = 2N$ for any N, given the gamma matrices in dimension $2N - 1$. The nice thing about this method is that it automatically gives the 'Weyl representation', in which $\hat{\gamma}$ is diagonal. This is easy to calculate explicitly (and I will give $\hat{\gamma}$ its traditional name, γ^5, in this dimension):

$$\gamma^5 = -\gamma^1\gamma^2\gamma^3\gamma^4 = \begin{pmatrix} -\mathrm{i}\sigma^1\sigma^2\sigma^3 & 0 \\ 0 & \mathrm{i}\sigma^1\sigma^2\sigma^3 \end{pmatrix} = \begin{pmatrix} \mathbf{1} & 0 \\ 0 & -\mathbf{1} \end{pmatrix}.$$

So with this representation of the gamma matrices, the top two components of a Dirac spinor correspond to a positive-chirality Weyl spinor, and the bottom two components correspond to a negative-chirality Weyl spinor.

Once again there are no Majorana spinors in this dimension.

2.4 $D = 6$

By now, we can easily write down the $D = 6$ gamma matrices in terms of what we already have. As a matter of notation, we will denote them by $\widetilde{\gamma}$ to distinguish them from their four-dimensional counterparts:

$$\widetilde{\gamma}^m = \begin{pmatrix} 0 & \mathrm{i}\gamma^m \\ -\mathrm{i}\gamma^m & 0 \end{pmatrix} \text{ for } m = 1, \ldots, 5 \; ; \quad \widetilde{\gamma}^6 = \begin{pmatrix} 0 & \mathbf{1} \\ \mathbf{1} & 0 \end{pmatrix},$$

where m runs over $1, \ldots, 5$, and $\mathbf{1}$ now represents the 4×4 identity. Once again, we have naturally obtained a Weyl representation; the chirality operator is

$$\widetilde{\gamma}^7 = \mathrm{i}\,\widetilde{\gamma}^1 \ldots \widetilde{\gamma}^6 = \begin{pmatrix} \mathbf{1} & 0 \\ 0 & -\mathbf{1} \end{pmatrix}.$$

Note that the generators σ^{mn}, projected onto either of the Weyl spinor representations, are a set of $\binom{6}{2} = 15$ anti-Hermitian 4×4 matrices, and therefore generate $SU(4)$. So we get another handy isomorphism $\mathrm{Spin}(6) \cong SU(4)$, and the two Weyl representations are the fundamental and anti-fundamental representations of this group.

This is the first time since $D = 2$ that we also have Majorana spinors. To see why, note that $\widetilde{\gamma}^m$ is real for $m = 1, 3, 6$, and purely imaginary for $m = 2, 4, 5$. We can therefore define charge conjugation by

$$\Psi^c = \widetilde{\gamma}^2\widetilde{\gamma}^4\widetilde{\gamma}^5\Psi^*,$$

and verify that $\left(\Psi^c\right)^c = \Psi$. So it is consistent to demand that $\Psi^c = \Psi$; the spinors satisfying this are of the form:

$$\Psi = \begin{pmatrix} \psi_R \\ \psi_L \end{pmatrix} = \begin{pmatrix} \psi_R \\ -\mathrm{i}\gamma^2\gamma^4\gamma^5\psi_R \end{pmatrix} = \begin{pmatrix} \psi_R \\ \mathrm{i}\gamma^1\gamma^3\psi_R^* \end{pmatrix} .$$

We can use this to change basis such that the matrices $\{\sigma^{mn}\}$ give a manifestly real (i.e. Majorana) representation of Spin(6). To do so, define $C = \mathrm{i}\gamma^1\gamma^3$, which satisfies $C^2 = 1$, and use this to define a unitary 8×8 matrix

$$M = \frac{1}{2}\begin{pmatrix} \mathbf{1} & C \\ -\mathrm{i}\,\mathbf{1} & \mathrm{i}\,C \end{pmatrix} .$$

If we act with this on a Majorana spinor, i.e. one satisfying $\Psi^c = \Psi$, we get

$$M\begin{pmatrix} \psi \\ C\psi^* \end{pmatrix} = \begin{pmatrix} \frac{1}{2}(\psi + \psi^*) \\ -\frac{\mathrm{i}}{2}(\psi - \psi^*) \end{pmatrix} = \begin{pmatrix} \Re(\psi) \\ \Im(\psi) \end{pmatrix} .$$

Now we consider the similarity transformation $\widetilde{\gamma}^m \mapsto M\widetilde{\gamma}^m M^{-1}$. The results are

$$\begin{aligned} &\widetilde{\gamma}^1 \mapsto \begin{pmatrix} -\gamma^3 & 0 \\ 0 & \gamma^3 \end{pmatrix} , \quad \widetilde{\gamma}^2 \mapsto \begin{pmatrix} 0 & -\mathrm{i}\gamma^4\gamma^5 \\ -\mathrm{i}\gamma^4\gamma^5 & 0 \end{pmatrix} , \quad \widetilde{\gamma}^3 \mapsto \begin{pmatrix} \gamma^1 & 0 \\ 0 & -\gamma^1 \end{pmatrix} , \\ &\widetilde{\gamma}^4 \mapsto \begin{pmatrix} 0 & \mathrm{i}\gamma^2\gamma^5 \\ \mathrm{i}\gamma^2\gamma^5 & 0 \end{pmatrix} , \quad \widetilde{\gamma}^5 \mapsto \begin{pmatrix} 0 & -\mathrm{i}\gamma^2\gamma^4 \\ -\mathrm{i}\gamma^2\gamma^4 & 0 \end{pmatrix} , \quad \widetilde{\gamma}^6 \mapsto \begin{pmatrix} \mathrm{i}\gamma^1\gamma^3 & 0 \\ 0 & -\mathrm{i}\gamma^1\gamma^3 \end{pmatrix} . \end{aligned} \tag{3}$$

Using the properties of the five-dimensional gamma matrices γ^m, it is easy to see that in this representation, the $\widetilde{\gamma}^m$ are still Hermitian, and now purely imaginary. For completeness, and for use later, we record the image of the chirality operator under this transformation:

$$\widetilde{\gamma}^7 \mapsto \begin{pmatrix} 0 & \mathrm{i}\,\mathbf{1} \\ -\mathrm{i}\,\mathbf{1} & 0 \end{pmatrix} , \tag{4}$$

which is also imaginary and Hermitian.

So if we use these gamma matrices, we can take our spinors to be simply real eight-dimensional vectors; in fact in this case the Majorana representation corresponds to the familiar embedding of $SU(4) \cong \mathrm{Spin}(6)$ in $SO(8)$.

2.5 $D = 8$

We now get to the big pay-off. In eight dimensions we can construct our gamma matrices, as usual, in terms of the seven dimensional ones. We will use the Majorana representation of these which was given in (3) and (4), and define 16×16 matrices

$$\Gamma^m = \begin{pmatrix} 0 & \mathrm{i}\widetilde{\gamma}^m \\ -\mathrm{i}\widetilde{\gamma}^m & 0 \end{pmatrix} \text{ for } m = 1, \dots, 7 \;\; ; \quad \Gamma^8 = \begin{pmatrix} 0 & \mathbf{1} \\ \mathbf{1} & 0 \end{pmatrix} .$$

Remembering that all the $\widetilde{\gamma}^m$ were purely imaginary, we see that these gamma matrices are real, and so we have obtained an $8D$ Majorana representation. But better still, it also splits naturally into two Weyl representations, given by the top eight and bottom eight components respectively. So in fact, these gamma matrices give us the two *Majorana-Weyl* representations in eight dimensions, which is something which does not exist in any lower dimension.

Note that, although the minimal spinors here have eight real components, we cannot choose the gamma matrices themselves to be 8×8 matrices, since they anti-commute with the chirality operator, and therfore exchange the two Weyl representations. If one wishes to work with just the eight-component spinors, the sort of formalism commonly used for two-component spinors of $SO(3,1)$ (and explained, for example, in Wess and Bagger's classic supersymmetry textbook) can easily be adapted to this case, starting from the matrices above.

3 Brief comments on Lorentzian signature

So far I have only discussed spinor representations of $SO(D)$, but spinors were originally discovered by Dirac in the Lorentzian case of $SO(3,1)$. Several things change in the indefinite signature case; let us focus on $SO(D-1,1)$ for concreteness, and because it is the most relevant case in physics.

The main difference now is that one gamma matrix squares to the negative of the identity, $(\gamma^0)^2 = -\mathbf{1}$. It therefore cannot be Hermitian; instead, we take γ^0 to be anti-Hermitian, and the remaining gamma matrices to be Hermitian, which is often summed up by the equation[1]

$$\gamma^{m\dagger} = \gamma^0 \gamma^m \gamma^0 \ . \tag{5}$$

The spin generators are still given by Equation (2), but now these are not all anti-Hermitian, so the representation is not unitary. Instead, it preserves the form defined by

$$\langle \chi, \psi \rangle = \bar{\chi}\psi \ ,$$

where $\bar{\chi} = \chi^\dagger \gamma^0$.

We can still define Weyl spinors when D is even, but Majorana and Majorana-Weyl spinors occur in different dimensions to the Euclidean signature case.

[1] In particle physics, the metric is often taken to have signature $(+1, -1, -1, -1)$, which would have the effect throughout of replacing Hermitian by anti-Hermitian and vice-versa, but (5) holds either way.